Journal of Computational and Applied Mathematics 41 (1992) 105–116
North-Holland

CAM 1160

Liouville–Green approximations for a class of linear oscillatory difference equations of the second order *

Renato Spigler

Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Università di Padova, Italy

Marco Vianello

Dipartimento di Matematica Pura e Applicata, Università di Padova, Italy

Received 25 January 1991
Revised 27 June 1991

Abstract

Spigler, R. and M. Vianello, Liouville–Green approximations for a class of linear oscillatory difference equations of the second order, Journal of Computational and Applied Mathematics 41 (1992) 105–116.

An asymptotic approximation theorem is proved for the solutions of linear oscillatory three-term recurrence equations in a certain class. This result represents a *discrete* analogue of the well-known Liouville–Green (or WKBJ) theorem rigorously proved by Olver for second-order linear *differential* equations. As in Olver's theorem, precise estimates are obtained for the error terms, and the *double asymptotic nature* of the approximation is made clear. Applications to certain orthogonal polynomials are also given.

Keywords: Liouville–Green approximation; linear difference equations; oscillatory solutions; orthogonal polynomials.

1. Introduction

Asymptotic theory for linear recurrence equations is important not only for combinatorics and computer science, but also for numerical analysis, orthogonal polynomials, continued fractions and many other areas of pure and applied mathematics. In a recent survey paper, Nevai pointed out that studying general techniques for deriving asymptotics for linear difference equations with variable coefficients is still an open research subject (cf. [12, Problem 2.28]). In [20], Wimp and Zeilberger have popularized a powerful method due to Birkhoff and

Correspondence to: Prof. R. Spigler, Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Università di Padova, via Belzoni 7, 35131 Padova, Italy.

* This work was supported, in part, by the Numerical Analysis "40%" funds, the Mathematical Analysis "60%" funds, and the GNFM-CNR.

Trjitzinsky [3,4], based on Poincaré's idea of asymptotic power series, to study solutions of n-term linear recurrent equations [15]. The original work by Birkhoff and Trjitzinsky was extremely massive and difficult, so that in [20] it has been clarified and then applied to several examples; see also [8,11] for other recent results.

In this paper, we present a first approach to a Liouville–Green (or WKBJ) theory for linear difference equations of the second order (three-term recurrence relations). There are several analogies but also important discrepancies between differential and difference equations, so that it is often difficult to carry certain properties or methods from the continuum into the discrete domain. The purpose of this work is to extend Olver's results for linear second-order oscillatory *differential* equations [13, Chapter 6] to the case of linear second-order oscillatory *difference* equations of the type

$$\Delta^2 y_n + q_n y_n = 0, \quad n \in \mathbb{N}_\nu, \tag{1.1}$$

Δ denoting the forward difference operator and

$$q_n = a + g_n, \quad a > 0, \tag{1.2}$$

a being independent of n, and g_n being any given sequence asymptotically small (in a suitable sense). In (1.1), $\mathbb{N}_\nu := \{n:\ n \in \mathbb{N},\ n \geqslant \nu\}$, where $\nu \in \mathbb{N}$, $\nu \geqslant 0$. Sometimes we shall use $\nu = -1$. The form (1.1) chosen here as the starting point for our analysis represents a kind of canonical (or reduced) form for general linear second-order difference equations, apart from an exceptional case (see below). Equation (1.1) is clearly analogous to the so-called Sturm–Liouville (or Jacobi) canonical form for differential equations. In fact, if one starts from

$$Y_{n+2} + A_n Y_{n+1} + B_n Y_n = 0, \quad n \in \mathbb{N}_\nu, \tag{1.3}$$

one obtains equation

$$\Delta^2 Y_n + a_n\, \Delta Y_n + b_n Y_n = 0, \quad n \in \mathbb{N}_\nu, \tag{1.4}$$

where

$$a_n = A_n + 2, \qquad b_n = A_n + B_n + 1. \tag{1.5}$$

Conversely, any equation like (1.4) can be transformed into (1.3) (with A_n, B_n given by (1.5)). Moreover, similarly to the case of differential equations, (1.4) can be transformed into (1.1) by setting

$$Y_n = \alpha_n y_n, \tag{1.6}$$

with

$$\alpha_n = \alpha_{\nu+1} \prod_{k=\nu}^{n-2} \left(1 - \tfrac{1}{2} a_k\right) = \alpha_{\nu+1} \prod_{k=\nu}^{n-2} \left(-\tfrac{1}{2} A_k\right), \quad n \geqslant \nu + 2, \tag{1.7}$$

(α_ν and $\alpha_{\nu+1} \neq 0$ being arbitrary constants), as it can be easily checked. Correspondingly, we obtain

$$q_n = -1 + \frac{4B_n}{A_n A_{n-1}} = -1 + \frac{4(b_n - a_n + 1)}{(a_n - 2)(a_{n-1} - 2)}, \quad n \geqslant \nu + 1, \tag{1.8}$$

$q_\nu = [\Delta^2 \alpha_\nu + (A_\nu + 2)\, \Delta\alpha_\nu + (A_\nu + B_\nu + 1)\alpha_\nu]/\alpha_{\nu+2}$. Clearly the special case $a_k = 2$ (i.e., $A_k =$

0) for infinitely many indices has to be ruled out. In any other case, one can use such a transformation for ν large enough, in particular to study the solutions of (1.3) asymptotically.

Other reduced forms do exist (again, with some exceptions), for instance, $\Delta^2 z_n + r_n z_{n+1} = 0$ (cf., e.g., [5]), or the self-adjoint form

$$\Delta(c_n \Delta z_n) + r_n z_{n+1} = 0 \tag{1.9}$$

(cf. [6,14]), which seems to be the most commonly used in oscillation theory.

In view of (2.1) below, equation (1.1) will be shown to be oscillatory, i.e., for every solution y_n and for any $N \in \mathbb{N}$ there is *an* integer $n \geqslant N$ such that $y_n y_{n+1} \leqslant 0$. This fact does not seem to be derivable directly from the most known oscillation criteria [14,21,22].

In Section 2, we state and prove the main theorem, which is a discrete version of the Liouville–Green approximation theorem proved by Olver for differential equations [13, Chapter 6], which applies to the class (1.1), (1.2). We emphasize that, as in [13], here the asymptotic results are accompanied by *precise estimates of the errors*. The leading idea is, like in [13], to base the error analysis on an associated Volterra integral equation. As a corollary, we can also prove a certain growth behavior that represents the discrete counterpart of a well-known result for differential equations (see [1]). In Section 3, finally, we give an application to orthogonal polynomials.

2. The main result

In this section we prove a theorem that extends [13, Theorem 2.2, p.196] with $f(x)$ (negative) constant to the discrete domain. Similarly to Olver's theorem, the quantity g_n in (1.2) can be complex-valued. In many applications, however, as well as within the classical oscillation theory, g_n is taken real-valued.

Theorem 2.1. *Consider* (1.1) *with* q_n *given by* (1.2), *where* $a > 0$ *and*

$$\sum_{n=\nu}^{\infty} |g_n| < \infty. \tag{2.1}$$

Then there exists $n_0 \in \mathbb{N}_\nu$ *such that there are two linearly independent solutions having the form*

$$y_n^{(j)} = (\lambda_j)^n (1 + \epsilon_n^{(j)}), \quad j = 1, 2, \ \textit{for } n \geqslant n_0, \tag{2.2}$$

where

$$\lambda_1 = 1 + i a^{1/2}, \qquad \lambda_2 = \bar{\lambda}_1 \tag{2.3}$$

are the roots of the characteristic polynomial associated to (1.1) *with* $g_n \equiv 0$, *and*

$$|\epsilon_n^{(j)}| \leqslant \frac{V_n}{1 - V_n}, \quad j = 1, 2, \quad n \geqslant n_0, \tag{2.4}$$

$$V_n := [a(1+a)]^{-1/2} \sum_{k=n}^{\infty} |g_k|. \tag{2.5}$$

If g_n *is real-valued, then* $y_n^{(1)}$ *and* $y_n^{(2)}$ *are complex conjugate.*

In the proof of Theorem 2.1, we shall see that n_0 can be taken as

$$n_0 = \min\{n: n \in \mathbb{N}_1, V_n < 1\}. \tag{2.6}$$

One should observe that, even disregarding the issue concerning the estimate of the errors, our asymptotic result could not be obtained by Perron's theorem (cf. [9, Theorem 3.3.2, p.75]), since $|\lambda_1| = |\lambda_2|$ in (2.3).

Before proving Theorem 2.1, we make some remarks that show differences and analogies with Olver's theorem. In (2.4), (2.5), V_n plays the role of the *variation* $V(x)$, and $\epsilon_n^{(j)}$ turns out to be of the order of V_n (since $V_n \to 0$) as $n \to \infty$. Moreover, every real solution has growing amplitude oscillations (at least on suitable subsequences), as $|\lambda_j| > 1$, $j = 1, 2$, other than the case $Y'' + (A + G(x))Y = 0$, $A = \text{const.} > 0$ (cf. [13, equation (7.01), p.208]), where the oscillations are bounded. As in [13], the *double asymptotic nature*, in n and a, is clearly visible, since $V_n \to 0$ when $n \to \infty$ *or* when $a \to \infty$. In particular $V_n = O(a^{-1})$ as $a \to \infty$, while $V(x) = O(A^{-1/2})$ in [13]. Finally, note that $g_n \to 0$ when $n \to \infty$, as a consequence of (2.1).

It is immediate to estimate $\epsilon_{n+1}^{(j)}$ and hence $\Delta\epsilon_n^{(j)}$ from (2.4), of course. The corresponding estimate for the derivative of the error terms in the continuous case is, on the other hand, nontrivial, and thus an essential part of Olver's theorem.

Proof of Theorem 2.1. Inserting (2.2) in (1.1), we get the "error equation"

$$\lambda^2 \Delta^2\epsilon_n + 2\lambda(\lambda - 1)\Delta\epsilon_n + g_n(1 + \epsilon_n) = 0, \tag{2.7}$$

where λ solves the characteristic equation $\lambda^2 - 2\lambda + (1 + a) = 0$, i.e., $\lambda = \lambda_j$, $j = 1, 2$, given by (2.3). Note that $y_n^{(j)}$ given by (2.2) is a solution to (1.1) if and only if $\epsilon_n^{(j)}$ solves (2.7). Hereafter, we shall study (2.7) with $\lambda = \lambda_1$, but the index 1 will be dropped for notational simplicity. We shall prove that any solution ϵ_n to the *discrete Volterra equation*

$$\epsilon_n = \frac{1}{2} \sum_{k=n}^{\infty} (1 - \mu^{k-n+1}) g_k \sigma (1 + \epsilon_k), \tag{2.8}$$

where

$$\mu = \frac{\lambda}{\bar{\lambda}}, \qquad \sigma = \frac{1}{\lambda(\lambda - 1)}, \tag{2.9}$$

is also a solution to (2.7). Moreover, we shall obtain a solution $\epsilon_n := \epsilon_n^{(1)}$, estimated as in (2.4). Notice that (2.8) is a discrete analogue of the integral equation appearing in [13, Chapter 6, §2.2, formula (2.09), p.194].

From (2.8) we obtain by easy calculations

$$\begin{aligned} \Delta\epsilon_n &= \frac{\mathrm{i}a^{1/2}}{\lambda} \sum_{k=n}^{\infty} \mu^{k-n+1} g_k \sigma (1 + \epsilon_k), \\ \Delta^2\epsilon_n &= -\frac{\mathrm{i}a^{1/2}}{\lambda} g_n \sigma (1 + \epsilon_n) - \frac{2\mathrm{i}a^{1/2}}{\lambda} \Delta\epsilon_n, \end{aligned} \tag{2.10}$$

where the recurrence $\Delta\mu^k = 2\mathrm{i}a^{1/2}\mu^{k+1}/\lambda$ (which follows from (2.9)) has been used. From (2.10) it follows immediately that (2.7) is satisfied.

Equation (2.8) can be solved by successive approximations. Below, we set for convenience $\epsilon(n) := \epsilon_n$ and define the sequence (of sequences) $\{\epsilon_s(n)\}_{s=0}^{\infty}$, as

$$\epsilon_0(n) \equiv 0, \qquad \epsilon_{s+1}(n) = \frac{1}{2} \sum_{k=n}^{\infty} (1 - \mu^{k-n+1}) g_k \sigma (1 + \epsilon_s(k)), \quad s = 0, 1, 2, \ldots . \tag{2.11}$$

One can immediately show (by induction on s) that all series in (2.11) converge for each fixed value of $n \in \mathbb{N}_\nu$, in view of (2.1) and of the fact that $|\mu| = 1$. Consider

$$\epsilon_{s+1}(n) - \epsilon_s(n) = \frac{1}{2} \sum_{k=n}^{\infty} (1 - \mu^{k-n+1}) g_k \sigma [\epsilon_s(k) - \epsilon_{s-1}(k)], \quad s \geqslant 1. \tag{2.12}$$

It is easy to prove inductively (on s) the estimate

$$|\epsilon_{s+1}(n) - \epsilon_s(n)| \leqslant (V_n)^{s+1}, \quad s = 0, 1, 2, \ldots, \quad n \in \mathbb{N}_\nu. \tag{2.13}$$

In fact, we have

$$|\epsilon_1(n)| \leqslant V_n, \tag{2.14}$$

as $|\mu| = 1$ and by using (2.1), (2.5). Assume now that (2.13) is satisfied with s replaced by $s - 1$. Then we get from (2.12)

$$|\epsilon_{s+1}(n) - \epsilon_s(n)| \leqslant \sum_{k=n}^{\infty} |g_k \sigma| (V_k)^s \leqslant (V_n)^{s+1}, \tag{2.15}$$

since $\{V_k\}$ is clearly a *decreasing* sequence (cf. (2.5)). At this point we introduce the sequence

$$\epsilon(n) := \sum_{s=0}^{\infty} [\epsilon_{s+1}(n) - \epsilon_s(n)], \tag{2.16}$$

which is well-defined in view of (2.15) whenever $V_n < 1$. This condition is satisfied for all $n \geqslant n_0$, since V_n decreases, n_0 being the smallest index not less than ν such that

$$\sum_{k=n_0}^{\infty} |g_k \sigma| < 1. \tag{2.17}$$

Moreover, we get from (2.13), (2.16),

$$|\epsilon(n)| \leqslant \frac{V_n}{1 - V_n}, \quad n \geqslant n_0. \tag{2.18}$$

Note, finally, that the series in (2.16) converges *uniformly* in n, for $n \geqslant n_0$.

The sequence $\epsilon(n)$ given by (2.16) solves the "integral equation" (2.8). In fact, we can write

$$\begin{aligned} \epsilon_n &= \epsilon_1(n) + \sum_{s=1}^{\infty} [\epsilon_{s+1}(n) - \epsilon_s(n)] \\ &= \frac{1}{2} \sum_{k=n}^{\infty} (1 - \mu^{k-n+1}) g_k \sigma + \frac{1}{2} \sum_{s=1}^{\infty} \sum_{k=n}^{\infty} (1 - \mu^{k-n+1}) g_k \sigma [\epsilon_s(k) - \epsilon_{s-1}(k)], \end{aligned} \tag{2.19}$$

and the proof will follow at once if we can show that the summation order in (2.19) can be interchanged. This can be justified by the Lebesgue *dominated convergence* theorem, as

$$\left|\frac{1}{2}\sum_{s=1}^{S}\left[\epsilon_s(k)-\epsilon_{s-1}(k)\right]g_k\sigma\left(1-\mu^{k-n+1}\right)\right| \leqslant |g_k\sigma|\frac{V_{n_0}}{1-V_{n_0}}, \tag{2.20}$$

for any $S>1$ and $k>n_0$, and using the summability condition (2.1).

If we take $\lambda=\lambda_2$ throughout the procedure outlined above, we get the same results with μ and σ replaced by their complex conjugate. In particular, as $\lambda_2=\bar{\lambda}_1$, if g_n is real-valued, we get $\epsilon_n^{(2)}=\overline{\epsilon_n^{(1)}}$. Moreover, $y_n^{(j)}=(\lambda_j)^n(1+\epsilon_n^{(j)})$, $j=1, 2$, are linearly independent. Indeed, their Casorati determinant [9, pp. 29–31]

$$C(n):=\begin{vmatrix} y_n^{(1)} & y_n^{(2)} \\ y_{n+1}^{(1)} & y_{n+1}^{(2)} \end{vmatrix} = (\lambda_1)^n(\lambda_2)^n\left[(\lambda_2-\lambda_1)+\mathrm{o}(1)\right], \quad n\to\infty, \tag{2.21}$$

is clearly nonzero for n sufficiently large, and thus for all $n\geqslant n_0$ (cf. [9, Theorem 2.1.2, p.29]). This completes the proof of Theorem 2.1. □

Remark 2.2. It is useful, at this point, to reformulate the hypotheses of Theorem 2.1 for (1.3). In this case, (1.2) and (2.1) become

$$\lim_{n\to\infty} q_n = a > 0, \qquad \sum_{n=\nu}^{\infty}|q_n-a|<\infty, \tag{2.22}$$

that is, by (1.8),

$$\lim_{n\to\infty}\frac{B_n}{A_nA_{n-1}} =: l > \frac{1}{4}, \tag{2.23}$$

where $l=\frac{1}{4}(a+1)$, and

$$\sum_{n=\nu+1}^{\infty}\left|\frac{B_n}{A_nA_{n-1}}-l\right|<\infty. \tag{2.24}$$

Remark 2.3. When g_n is a *real* sequence, the pair

$$\mathrm{Re}\; y_n^{(1)}=\rho^n(\cos n\theta+\xi_n), \qquad \mathrm{Im}\; y_n^{(1)}=\rho^n(\sin n\theta+\tau_n), \tag{2.25}$$

where

$$\rho:=|\lambda_1|, \qquad \theta:=\arg(\lambda_1), \tag{2.26}$$

and

$$\xi_n:=\mathrm{Re}\,\epsilon_n^{(1)}\cos n\theta-\mathrm{Im}\,\epsilon_n^{(1)}\sin n\theta, \qquad \tau_n:=\mathrm{Re}\,\epsilon_n^{(1)}\sin n\theta+\mathrm{Im}\,\epsilon_n^{(1)}\cos n\theta \tag{2.27}$$

clearly represent a basis for the real solutions to (1.1). Moreover, similarly to [13, Example 2.5, p.197], we can write the general solution $y_n=\alpha\,\mathrm{Re}\, y_n^{(1)}+\beta\,\mathrm{Im}\, y_n^{(1)}$ in the form

$$y_n=A\rho^n\left[\cos(n\theta+\delta)+\left|\epsilon_n^{(1)}\right|\cos(n\theta+\gamma_n)\right], \tag{2.28}$$

where

$$A=\left(\alpha^2+\beta^2\right)^{1/2}, \qquad \mathrm{tg}\;\delta=-\frac{\beta}{\alpha}, \qquad \mathrm{tg}\;\gamma_n=\frac{\alpha\,\mathrm{Im}\,\epsilon_n^{(1)}-\beta\,\mathrm{Re}\,\epsilon_n^{(1)}}{\alpha\,\mathrm{Re}\,\epsilon_n^{(1)}+\beta\,\mathrm{Im}\,\epsilon_n^{(1)}}. \tag{2.29}$$

It is interesting to establish, as a consequence of Theorem 2.1, the following corollary.

Corollary 2.4. *Under the hypotheses of Theorem* 2.1, *any solution* y_n *to* (1.1) *is subject to the growth estimate*

$$|y_n| \leqslant C|\lambda_1|^n, \quad n \geqslant n_0, \tag{2.30}$$

where C is a constant depending on the particular choice of the solution $\{y_n\}$ (*and on* n_0).

The proof follows immediately from (2.2).

Remark 2.5. From (1.6) and (2.30) the estimate

$$|Y_n| \leqslant C|\alpha_n||\lambda_1|^n, \quad n \geqslant n_0, \tag{2.31}$$

follows, valid for any solution Y_n of (1.3) that can be transformed into (1.1) by (1.6), (1.7).

Corollary 2.4 can be regarded as the discrete counterpart of a well-known *boundedness* result for the solutions to second-order differential equations like $y'' + [1 + g(x)]y = 0$ with $g \in L^1(c, +\infty)$, due to Ascoli (cf. [1], [2, p.112]; see also [16, p.162]). Note that the summability of g_n is the key condition in Theorem 2.1.

3. An application to orthogonal polynomials

The class $\{q_n\}$ considered in this paper is sufficiently broad to include several important examples. In fact, it is well known that, e.g., all orthogonal polynomials on the real line with respect to a positive Borel measure satisfy a three-term linear recurrence relation such as, for instance,

$$P_{n+2}(x) - (\mathscr{A}_n x + \mathscr{B}_n)P_{n+1}(x) + C_n P_n(x) = 0 \tag{3.1}$$

(unrestricted form), or

$$P_{n+2}(x) + (c_n - x)P_{n+1}(x) + \lambda_n P_n(x) = 0 \tag{3.2}$$

(monic form), where $P_{-1}(x) \equiv 0$, $P_0(x) \equiv 1$, $\mathscr{A}_n \mathscr{A}_{n-1} C_n > 0$, cf. [7, §4].

Now, we show that certain families of orthogonal polynomials can be studied asymptotically for $n \to \infty$ (x being considered as a fixed parameter) by the technique developed in Section 2. Obtaining asymptotic representations has always been recognized as one of the finest problems in the theory of orthogonal polynomials [10, p.303]. Deriving such representations from the recurrence relation is at the present time subject of extensive and deep investigations (cf. [8,11,12,19] and the references quoted therein). The examples worked out below represent just a first attempt to use the discrete Liouville–Green approximation to obtain *qualitative* information on orthogonal polynomials starting from their recurrence relation.

Example 3.1 (*Legendre polynomials*). Legendre polynomials satisfy (1.3) with

$$A_n = \mathscr{A}_n x + \mathscr{B}_n = -\frac{2n+3}{n+2}x, \qquad B_n = C_n = \frac{n+1}{n+2}, \tag{3.3}$$

cf., e.g., [17, formula (4.7.17), p.82] with $\lambda = \frac{1}{2}$. Therefore we get from (1.8),

$$q_n(x) = -1 + \frac{4}{x^2}\frac{(n+1)^2}{(2n+3)(2n+1)}. \tag{3.4}$$

In (3.4), we must take $x \neq 0$, of course. In fact, the transforming function $\alpha_n \equiv \alpha_n(x)$ in (1.6), (1.7) is "singular" at $x = 0$, and we get (for $\nu = -1$, $\alpha_0 = 1$)

$$\alpha_n(x) = \frac{(2n-1)!!}{(2n)!!}\left(\tfrac{1}{2}x\right)^n. \tag{3.5}$$

Therefore,

$$\lim_{n\to\infty} q_n(x) = -1 + \frac{1}{x^2} =: a(x) \quad \text{(finite and } > 0 \text{ for } 0 < |x| < 1\text{)}, \tag{3.6}$$

$$\sum_{n=-1}^{\infty} |q_n(x) - a(x)| = \frac{1}{x^2}\sum_{n=-1}^{\infty}\frac{1}{(2n+3)(2n+1)} < \infty. \tag{3.7}$$

We then obtain

$$\begin{aligned}
\lambda \equiv \lambda_1(x) &= 1 + \mathrm{i}\frac{(1-x^2)^{1/2}}{|x|} =: \rho\, \mathrm{e}^{\mathrm{i}\theta} = \frac{1}{|x|}\mathrm{e}^{\mathrm{i}\theta(x)},\\
\mu \equiv \mu(x) &= \frac{|x| + \mathrm{i}(1-x^2)^{1/2}}{|x| - \mathrm{i}(1-x^2)^{1/2}},\\
\sigma \equiv \sigma(x) &= \frac{x^2}{\mathrm{i}(1-x^2)^{1/2}\left[|x| + \mathrm{i}(1-x^2)^{1/2}\right]},
\end{aligned} \tag{3.8}$$

where $0 < |x| < 1$. Note that $\mu(x)$ and $\sigma(x)$ can be continued up to $x = 0$. By (1.6) and (2.2) we are able to represent a *basis* for the solutions to (3.1), (3.3) for each fixed x, $0 < |x| < 1$:

$$\begin{aligned}
Y_n^{(j)}(x) &= \alpha_n(x)\left[\lambda_j(x)\right]^n\left[1 + \epsilon_n^{(j)}(x)\right]\\
&= (\operatorname{sgn} x)^n \frac{(2n-1)!!}{(2n)!!}\exp\{(-1)^{j+1}\mathrm{i}n\theta(x)\}\left[1 + \epsilon_n^{(j)}(x)\right],\\
&\qquad j = 1, 2,\ n \geqslant n_0(x).
\end{aligned} \tag{3.9}$$

Here $n_0(x)$ is the smallest index such that $\sum_{j=n_0(x)}^{\infty} |\sigma(x) g_j(x)| < 1$ (cf. (2.6)), and

$$|\epsilon_n^{(j)}(x)| \leqslant \frac{V_n(x)}{1 - V_n(x)}, \quad V_n(x) = (1-x^2)^{-1/2}\sum_{k=n}^{\infty}\frac{1}{(2k+3)(2k+1)} = \frac{(1-x^2)^{-1/2}}{2(2n+1)}, \tag{3.10}$$

for $n \geqslant n_0(x)$, cf. (2.4), (2.5). Note that $V_n(x) = O(n^{-1})$ as $n \to \infty$. Clearly, $n_0(x)$ and the estimate (3.10) can be given *uniformly* in x, for x in an arbitrary but fixed compact subset of $(-1, 1)$. One could show, at this point, that the error terms $\epsilon_n^{(j)}(x)$ are *continuous* functions of

x, for $x \in K$, K being any compact subset of $(-1, 1)$, for all $n \geqslant n_0$. The proof can be based on the uniform convergence of the series in (2.11) and (2.16) on K. Details are left to the reader.

The basis functions $Y_n^{(j)}(x)$ are, however, discontinuous at $x = 0$ for all odd n. Thus, the nth Legendre polynomial can be written as

$$P_n(x) = c_1(x)Y_n^{(1)}(x) + c_2(x)Y_n^{(2)}(x), \quad n \geqslant n_0(x), \tag{3.11}$$

where $c_2 = \overline{c_1}$ since $Y_n^{(2)} = \overline{Y_n^{(1)}}$. The combinators $c_j(x)$, $j = 1, 2$, can be obtained by inverting the system formed by (3.11) and the same equation with n replaced by $n + 1$. They now appear as continuous functions for $0 < |x| < 1$ as a consequence of the continuity of $\epsilon_n^{(j)}(x)$ and of the fact that the *Casorati determinant* of $Y_n^{(1)}$, $Y_n^{(2)}$,

$$\begin{aligned} C\left[Y_n^{(1)}, Y_n^{(2)}\right] &= Y_n^{(1)}Y_{n+1}^{(2)} - Y_n^{(2)}Y_{n+1}^{(1)} \\ &= \operatorname{sgn} x \frac{(2n-1)!!}{n!} \frac{(2n+1)!!}{(n+1)!} \frac{1}{2^{2n}} \left[-\mathrm{i} \sin\theta + \mathrm{O}\left(\epsilon_n^{(j)}\right)\right] \end{aligned} \tag{3.12}$$

is a continuous nonvanishing function for $0 < |x| < 1$ (cf. (3.9) and (2.21)).

By using (2.28), we can represent $P_n(x)$ as

$$\begin{aligned} P_n(x) &= A(x)\alpha_n(x)\rho^n(x)[\cos(n\theta(x) + \delta(x)) + E_n(x)] \\ &= (\operatorname{sgn} x)^n \frac{(2n-1)!!}{(2n)!!} A(x)[\cos(n\theta(x) + \delta(x)) + E_n(x)], \quad n \geqslant n_0(x), \end{aligned} \tag{3.13}$$

since $\rho(x) = 1/|x|$, $\theta(x) = \arg(\lambda_1(x))$ (cf. (3.8)), where $A(x)$ and $\delta(x)$ can be immediately related to $c_j(x)$, $j = 1, 2$. In particular, they also are continuous functions of x for $0 < |x| < 1$. We stress again that our approach is mainly *qualitative*. Using (3.13), for instance, we get

$$|P_n(x)| \leqslant \frac{(2n-1)!!}{(2n)!!} \frac{|A(x)|}{1 - V_n(x)} = \mathrm{O}(n^{-1/2}) \tag{3.14}$$

(cf. (2.31)), uniformly valid on compact subsets of $0 < |x| < 1$.

Deriving explicitly the c_j's (or A and δ) is possible, in general, proceeding numerically, e.g., by evaluating all polynomials $P_1(x), \ldots, P_m(x)$ for a suitable $m \geqslant n_0 + 1$, and then solving the linear system obtained from (3.11) for $n = m$ and $n = m - 1$. In this procedure, symbolic manipulations may be useful. The problem of identifying or approximating unknown parameters affects Birkhoff's method as well, cf. [20, Example 3.3, p.175]. In the present case, the lowest-order term in the well-known Darboux asymptotic formula

$$P_n(\cos\phi) = \left(\frac{2}{\sin\phi}\right)^{1/2} \frac{(2n-1)!!}{(2n)!!} \cos\left[\left(n + \tfrac{1}{2}\right)\phi - \tfrac{1}{4}\pi\right] + \mathrm{O}(n^{-3/2}), \tag{3.15}$$

where $x = \cos\phi$, $0 < \phi < \pi$ (cf. [17, formula (8.21.4), p.189]), allows us to identify the coefficients $A(x)$ and $\delta(x)$. In fact, (3.13) can be rewritten as

$$P_n(x) = (\operatorname{sgn} x)^n \frac{(2n-1)!!}{(2n)!!} A(x) \cos[n\theta(x) + \delta(x)] + \mathrm{O}(n^{-3/2}), \tag{3.16}$$

since $(2n-1)!!/(2n)!!=O(n^{-1/2})$ and $E_n=O(n^{-1})$ (by (2.28), (3.10), (3.13)). Suppose first $x>0$. Then, clearly, $\theta=\phi$. Multiplying by $n^{1/2}$ both sides of (3.15) and (3.16) and subtracting, we get

$$\left(\frac{2}{\sin\phi}\right)^{1/2}\cos\left[\left(n+\tfrac{1}{2}\right)\phi-\tfrac{1}{4}\pi\right]-A\cos(n\phi+\delta)=o(1), \tag{3.17}$$

as $n\to\infty$. It follows easily that

$$|A(x)|=\left(\frac{2}{\sin\phi}\right)^{1/2}=\frac{\sqrt{2}}{(1-x^2)^{1/4}},\qquad \delta(x)=\tfrac{1}{2}\phi(x)-\tfrac{1}{4}\pi\ (\mathrm{mod}\ \pi). \tag{3.18}$$

Since δ in (3.13) *is* defined up to a multiple of π (cf. (2.28)), choosing $\delta=\frac{1}{2}\phi-\frac{1}{4}\pi$ we get $A(x)>0$. If $x<0$, $\theta=\pi-\phi$ and easy calculations yield the same $A(x)$ and $\delta(x)=\frac{1}{4}\pi-\frac{1}{2}\phi(x)$.

Therefore, Darboux' asymptotic formula (3.15) has been re-obtained, along *with* an upper bound for the *absolute error* for $n\geqslant n_0(x)$ (the *relative* error was already given in terms of $E_n(x)$ in (3.13)):

$$\frac{(2n-1)!!}{(2n)!!}|A(x)|\,|E_n(x)|\leqslant\frac{\sqrt{2}}{(1-x^2)^{1/4}}\frac{(2n-1)!!}{(2n)!!}\frac{1}{2(2n+1)(1-x^2)^{1/2}-1} \tag{3.19}$$

(cf. (3.10)). Again, such an estimate holds uniformly on compact subsets of $(-1,1)$.

Example 3.2 (*Pollaczek polynomials*). We can show that our method applies to a subfamily of Pollaczek polynomials that contains the ultraspherical polynomials. Pollaczek polynomials $P_n^\lambda(x;\,a,\,b,\,c)$ satisfy (1.3) with

$$A_n=-2\frac{(n+\lambda+a+c+1)x+b}{n+c+2},\qquad B_n=\frac{n+2\lambda+c}{n+c+2} \tag{3.20}$$

(cf. [7, (5.8), p.185]). The parameters a, b, c, λ are subject to the limitations $a\geqslant|b|$, $2\lambda+c>0$, $c\geqslant0$, *or* $a\geqslant|b|$, $2\lambda+c\geqslant1$, $c>-1$. It follows that the transformation

$$\alpha_n(x)=\prod_{k=-1}^{n-2}\frac{(k+\lambda+a+c+1)x+b}{k+c+2} \tag{3.21}$$

is well-defined. The points where $\alpha_n(x)=0$ can be easily handled when $b\neq0$ (the transformation can be considered for ν sufficiently large, cf. (1.7)); when $b=0$, $x=0$ is indeed a singular point of the transformation.

It is immediate to see that $q_n\to-1+1/x^2=:a(x)>0$ for $0<|x|<1$, as in Example 3.1, q_n being defined by (1.8). Now, it is easy to check that $\sum_{k=-1}^{\infty}|q_k-a|$ converges if and only if $a=b=0$. Therefore, our method applies to the subfamily $P_n^\lambda(x;\,0,\,0,\,c)$ with $2\lambda+c>0$, $c\geqslant0$, *or* $2\lambda+c\geqslant1$, $c>-1$. We obtain

$$g_n(x)-q_n(x)-a(x)=\frac{\lambda(1-\lambda)}{x^2(n+\lambda+c)(n+\lambda+c+1)}. \tag{3.22}$$

Moreover, $\lambda_1(x)$, $\mu(x)$ and $\sigma(x)$ are *the same* functions as those obtained in Example 3.1 (cf. (3.8)), and

$$V_n(x) = \frac{|\lambda||1-\lambda|}{(1-x^2)^{1/2}} \sum_{k=n}^{\infty} \frac{1}{|(k+\lambda+c)(k+\lambda+c+1)|}. \tag{3.23}$$

For $a=b=0$, (3.21) yields

$$\alpha_n(x) = x^n \prod_{k=-1}^{n-2} \frac{k+\lambda+c+1}{k+c+2} = \frac{(\lambda+c)_n}{(c+1)_n} x^n = \frac{\Gamma(n+\lambda+c)}{\Gamma(n+c+1)} \frac{\Gamma(c+1)}{\Gamma(\lambda+c)} x^n, \quad x \neq 0, \tag{3.24}$$

$(z)_n$ denoting the Pochhammer symbol. Therefore, a representation holds for $n \geqslant n_0(x)$ like (3.9), for two linearly independent solutions of (1.3), (3.20). The error term $\epsilon_n^{(j)}(x)$ is estimated as in (2.4) where $V_n(x)$ is given by (3.23). Moreover, we get for $n \geqslant n_0(x)$

$$P_n^{\lambda}(x; 0, 0, c) = (\operatorname{sgn} x)^n \frac{(\lambda+c)_n}{(c+1)_n} A(x)\{\cos[n\theta(x) + \delta(x)] + E_n(x)\}, \tag{3.25}$$

where $|E_n(x)| \leqslant |\epsilon_n^{(1)}(x)|$ and again $A(x)$ and $\delta(x)$ are continuous functions for $0 < |x| < 1$ (cf. Example 3.1). Standard calculations yield the estimate

$$|P_n^{\lambda}(x; 0, 0, c)| \leqslant \left| \frac{(\lambda+c)_n}{(c+1)_n} \right| \frac{A(x)}{1-V_n(x)} = \mathrm{O}(n^{\lambda-1}), \quad n \geqslant n_0(x), \tag{3.26}$$

(cf. (3.14) (recall that $P_n^{1/2}(x; 0, 0, 0)$ are the Legendre polynomials) and Remark 2.5).

The special class of Pollaczek polynomials with $a=b=c=0$ coincides with the ultraspherical polynomials. Proceeding as in Example 3.1, we can identify $A(x)$ and $\delta(x)$ in (3.25) with $c=0$, comparing (3.25) with

$$P_n^{(\lambda)}(\cos\phi) = \frac{\left(\frac{1}{2}n\right)^{\lambda-1}}{\Gamma(\lambda)(\sin\phi)^{\lambda}} \left\{\cos\left[(n+\lambda)\phi - \tfrac{1}{2}\lambda\pi\right] + \mathrm{O}(n^{-1})\right\}, \tag{3.27}$$

$0 < \phi < \pi$ (cf. [18, formula (7.25), p.183]). The result is

$$A(x) = \frac{2^{1-\lambda}}{(1-x^2)^{\lambda/2}}, \qquad \delta(x) = \begin{cases} \lambda\phi - \frac{1}{2}\lambda\pi, & x > 0, \\ \frac{1}{2}\lambda\pi - \lambda\phi, & x < 0, \end{cases} \tag{3.28}$$

(cf. formula (3.18) and the remarks following it).

References

[1] G. Ascoli, Sul comportamento asintotico delle equazioni differenziali del 2° ordine, *Rend. Lincei* (6) **22** (1935) 234–243.

[2] R. Bellman, *Stability Theory of Differential Equations* (McGraw-Hill, New York, 1953).

[3] G.D. Birkhoff, Formal theory of irregular difference equations, *Acta Math.* **54** (1930) 205–246.

[4] G.D. Birkhoff and W.J. Trjitzinsky, Analytic theory of singular difference equations, *Acta Math.* **60** (1932) 1–89.

[5] S.S. Cheng, Monotone solutions of $\Delta^2 x(k) = Q(k)x(k+1)$, *Chinese J. Math.* **10** (1982) 71–75.

[6] S.S. Cheng, Sturmian comparison theorems for three-term recurrence equations, *J. Math. Anal. Appl.* **111** (1985) 465–474.

[7] T.S. Chihara, *An Introduction to Orthogonal Polynomials* (Gordon and Breach, New York, 1978).

[8] S.W. Jha, A. Máté and P. Nevai, Asymptotics for the solutions of systems of smooth recurrence equations and their applications to orthogonal polynomials, in: K.C. Charles, L.L. Schumaker and J.D. Ward, Eds., *Approximation Theory VI* (Academic Press, San Diego, CA, 1989) 345–348.

[9] V. Lakshmikantham and D. Trigiante, *Theory of Difference Equations: Numerical Methods and Applications* (Academic Press, San Diego, CA, 1988).

[10] D.S. Lubinsky, The approximate approach to orthogonal polynomials for weights on $(-\infty, \infty)$, in: P. Nevai, Ed., *Orthogonal Polynomials: Theory and Practice*, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. **294** (Kluwer Academic Publishers, Dordrecht, 1990) 293–310.

[11] A. Máté and P. Nevai, Asymptotics for solutions of smooth recurrence equations, *Proc. Amer. Math. Soc.* **93** (1985) 423–429.

[12] P. Nevai, Research problems in orthogonal polynomials, in: K.C. Charles, L.L. Schumaker and J.D. Ward Eds., *Approximation Theory VI* (Academic Press, San Diego, CA, 1989) 449–489.

[13] F.W.J. Olver, *Asymptotics and Special Functions* (Academic Press, New York, 1974).

[14] W. Patula, Growth and oscillation properties of second order linear difference equations, *SIAM J. Math. Anal.* **10** (1979) 55–61.

[15] H. Poincaré, Sur les équations linéaires aux différentielles et aux différences finies, *Amer. J. Math.* **7** (1885) 203–258.

[16] D.R. Smith, Liouville–Green approximations via the Riccati transformation, *J. Math. Anal. Appl.* **116** (1986) 147–165.

[17] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. **23** (Amer. Mathematical Soc., New York, 1939).

[18] F.G. Tricomi, *Vorlesungen über Orthogonalreihen* (Springer, Berlin, 1955).

[19] W. Van Assche, Asymptotics for orthogonal polynomials and three-term recurrences, in: P. Nevai, Ed., *Orthogonal Polynomials: Theory and Practice*, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. **294** (Kluwer Academic Publishers, Dordrecht, 1990) 435–462.

[20] J. Wimp and D. Zeilberger, Resurrecting the asymptotics of linear recurrences, *J. Math. Anal. Appl.* **111** (1985) 162–176.

Added in proof:

[21] R.P. Agarwal, *Difference Equations and Inequalities* (Marcel Dekker, New York, 1991).

[22] W.G. Kelley and A.C. Peterson, *Difference Equations* (Academic Press, San Diego, CA, 1991).